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THE ALGEBRAIC STRUCTURE AND POISSON'S THEORY FOR THE EQUATIONS OF MOTION OF NON-HOLONOMIC SYSTEMS†

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Poisson's theory for holonomic conservative systems is shown to apply partially to non-holonomic systems.

Existing methods of integrating the equations of non-holonomic dynamics are based on the laws of conservation [1], reducing the order of the equations [2], Noether's theorem [3, 4], an extension of the Hamilton–Jacobi method [5], a field method [6, 7], the method of invariant measure [8], etc.

Poisson's theory for holonomic conservative systems makes use of the fact that the Hamiltonian equations possess an algebraic structure, whence it follows, in particular, that the Poisson bracket can be introduced for non-holonomic systems and Poisson's theory can be partly extended to them.

1. Let the position of a mechanical system be defined by the generalized coordinates q_s ($s = 1, \ldots, n$) and the system is under ideal non-holonomic constraints (as stated by Chetayev)

$$f_\beta(q_s, \dot{q}_s, t) = 0, \quad \beta = 1, \ldots, g \tag{1.1}$$

We write the equations of motion of the system in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s + \Lambda_s, \quad \Lambda_s = \Lambda_s(q, \dot{q}, t) = \sum_{\beta=1}^{g} \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \tag{1.2}$$

Let

$$\tilde{Q}_s = Q_s + \Lambda_s = Q_s' + Q_s'', \quad Q_s' = \frac{\partial U}{\partial q_s} - \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_s}$$

Then Eqs (1.2) take the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s'', \quad L = T + U \tag{1.3}$$

These equations can be used to describe a holonomic system with n degrees of freedom, with generalized forces Q_s'' and Lagrange function L. If the initial conditions satisfy Eqs (1.1), the solution of Eq. (1.3) gives the motion of the non-holonomic system (1.1), (1.2) [9].

2. We will consider the algebraic structure of Eqs (1.3). Let

$$p_s = \frac{\partial L}{\partial \dot{q}_s}, \quad H = \sum_{s=1}^{n} p_s \dot{q}_s - L \tag{2.1}$$

$$Q_s'' = -\sum_{k=1}^{n} \Omega_{sk} \frac{\partial H}{\partial p_k}, \quad \Omega_{sk} = \operatorname{diag}(\Omega_{11}, \ldots, \Omega_{nn}) \tag{2.2}$$

Equations (1.3) take the form

$$\dot{a}^\mu - S^{\mu\nu} \frac{\partial H}{\partial a^\nu} = 0, \quad \mu, \nu = 1, \ldots, 2n \tag{2.3}$$

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$$a^{\mu}=\begin{cases}q_{\mu}, & \mu=1,\dots,n\\ p_{\mu-n}, & \mu=n+1,\dots,2n\end{cases},\quad S^{\mu\nu}=\omega^{\mu\nu}+T^{\mu\nu}$$

$$\omega^{\mu\nu}=\begin{Vmatrix}O_{n\times n} & I_{n\times n}\\ -I_{n\times n} & O_{n\times n}\end{Vmatrix},\quad T^{\mu\nu}=\begin{Vmatrix}O_{n\times n} & O_{n\times n}\\ O_{n\times n} & (-\Omega_{kk})_{n\times n}\end{Vmatrix}$$

Summation over repeated indices is carried out everywhere.

We will represent the derivative with respect to time of a function $A(a^{\mu})$ in the form of the algebraic product

$$\dot{A}=\frac{\partial A}{\partial a^{\mu}}S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\triangleq A\circ H \tag{2.4}$$

This product possesses the following properties

$$A\circ(B+C)=A\circ B+A\circ C,\quad (A+B)\circ C=A\circ C+B\circ C$$
$$(\alpha A)\circ B=A\circ(\alpha B)=\alpha(A\circ B)$$

Hence it possesses an intrinsic algebraic structure. It does not, however, provide a Lie algebra in the general case.

We define the new product

$$[A,B]\triangleq A\circ B-B\circ A \tag{2.5}$$

which is skew-symmetric and satisfies the Jacobi identity

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$$

Hence the equations of motion (1.3) possess a structure for which there is a Lie algebra. In the special case where $T^{\mu\nu}=0$, the equations of motion (1.3) possess the algebraic structure of a Lie algebra, that is, they are Hamiltonian.

3. We now integrate the equations of motion (1.3) on the basis of their algebraic structure and Poisson's theory. Suppose that $I(a^{\mu}, t)$ = const is the first integral of Eqs (2.3). Then, using definition (2.4), we have the identity

$$\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^{\mu}}S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}=\frac{\partial I}{\partial t}+I\circ H=0 \tag{3.1}$$

which is the generalized Poisson condition for the integral of Eqs (2.3).

Substituting $I = H$ into (3.1) we obtain

$$\frac{\partial H}{\partial t}+H\circ H=\frac{\partial H}{\partial t}+\frac{\partial H}{\partial a^{\mu}}T^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}=\frac{\partial H}{\partial t}+\lambda_{\beta}\frac{\partial f_{\beta}}{\partial \dot{q}_s}\dot{q}_s \tag{3.2}$$

If

$$\frac{\partial H}{\partial t}=0,\quad \frac{\partial f_{\beta}}{\partial \dot{q}_s}=\dot{q}_s k_{\beta} f_{\beta} \tag{3.3}$$

the right-hand side of Eq. (3.2) vanishes. We thus have the following theorem [10].

Theorem 1. If the Hamilton function H is independent of time and the equations of non-holonomic constraints (1.1) are homogeneous with respect to generalized velocities, then the Hamilton function will be the first integral of system (2.3).

Taking the partial derivative with respect to t of both sides of Eq. (3.1) in the case where $S^{\mu\nu}$ and H are independent of t, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial t}\right)+\frac{\partial I}{\partial t}\circ H=0$$

A similar equation holds for $\partial^2 I/\partial t^2$. $\partial^3 I/\partial t^3, \dots$. We thus have the following theorem.

Theorem 2. If system (2.3) has a first integral which depends on time, and H and $S^{\mu\nu}$ are independent of time, then $\partial I/\partial t$, $\partial^2 I/\partial t^2$, $\partial^3 t/\partial t^3 \dots$ are first integrals of system (2.3).

The next theorem can be proved in the same way by taking the partial derivative with respect to a^p of both sides of Eq. (3.1).

Theorem 3. If system (2.3) has a first integral which contains a^p, and H and $S^{\mu\nu}$ do not contain a^p, then $\partial I/\partial a^p$, $\partial^2 I/\partial a^{p2}$, $\partial^3 I/\partial a^{p3}$ are first integrals of system (2.3).

In the general case, the product (2.4) does not form a Poisson bracket or generalized Poisson bracket and Poisson's theory cannot be applied to system (2.3) in the general case.

4. *Examples of the use of Theorems* 1–3. In Appell's example the Lagrange function and constraint equation have the form

$$L = \tfrac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^3) - mgq_3, \quad C^2\dot{q}_3^2 = \dot{q}_1^2 + \dot{q}_2^2$$

Equations (1.3) have the form

$$m\ddot{q}_1 = -\xi C^2\dot{q}_3\dot{q}_1, \quad m\ddot{q}_2 = -\xi C^2\dot{q}_3\dot{q}_2, \quad m\ddot{q}_3 = -mg + \xi C^4\dot{q}_3^2$$
$$\xi = mg/(\dot{q}_1^2 + \dot{q}_2^2 + C^2\dot{q}_3^2)$$

We have

$$p_s = \partial L/\partial \dot{q}_s, \quad s = 1,2,3$$
$$H = \sum_{s=1}^{3} p_s\dot{q}_s - L = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + mgq_3$$

Then, using identity (2.2), we obtain

$$\Omega_{11} = \eta C^2 p_3, \quad \Omega_{22} = \eta C^2 p_3, \quad \Omega_{33} = -\eta C^4 p_3$$
$$\eta = m^2 g/(p_1^2 + p_2^2 + C^4 p_3^2)$$

We put $n = 3$ in Eqs (2.3). It follows from Theorem 1 that the Hamilton function H will be a first integral

$$H = \frac{1}{2m}\{(a^4)^2 + (a^5)^2 + (a^6)^2\} + mga^3 = h = \text{const}$$

Note that this system has the first integral

$$I_1 = \{a^1 - (a^4/a^5)a^2\}^2 = \text{const}$$

and H and $S^{\mu\nu}$ do not contain a^1, a^2. By Theorem 3, we obtain the first three integrals

$$I_2 = \partial I_1/\partial a^1 = \{a^1 - (a^4/a^5)a^2\} = \text{const}$$
$$I_3 = \partial I_1/\partial a^2 = 2\{a^1 - (a^4/a^5)a^2\}(-a^4/a^5) = \text{const}$$
$$I_4 = \partial^2 I_1/\partial a^1\partial a^2 = -2a^4/a^5 = \text{const}$$

We note that this system has the first integral

$$I_5 = (a^6 + \zeta t)^2 = \text{const}, \quad \zeta = g/(C^2 + 1)$$

and H and $S^{\mu\nu}$ are independent of t. Using Theorem 2, we obtain the first integral

$$I_6 = \partial I_5/\partial t = 2(a^6 + \zeta t)\zeta = \text{const}$$

In the example of Novoselev [9], the Lagrange function and equation of non-holonomic constraint have the form

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2), \quad f = \dot{q}_1 + bt\dot{q}_2 - bq_2 + t = 0$$

The equations of motion (1.3) give

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}'_s} - \frac{\partial L'}{\partial q_s} = 0, \quad s = 1,2$$

$$L' = L + \frac{1}{b}\dot{q}_1 \operatorname{arctg} bt + \frac{1}{2b}\dot{q}_2 \ln(1 + b^2 t^2)$$

In Eqs (2.3) we put $n = 2$, where

$$a^1 = q_1, \quad a^2 = q_2, \quad a^3 = p_1 = \dot{q}_1 + \frac{1}{b}\operatorname{arctg} bt$$

$$a^4 = p_2 = \dot{q}_2 + \frac{1}{2b}\ln(1 + b^2 t^2)$$

$$H = \frac{1}{2}\left[p_1 - \frac{1}{b}\operatorname{arctg} bt\right]^2 + \frac{1}{2}\left[p_2 - \frac{1}{2b}\ln(1 + b^2 t^2)\right]^2$$

Poisson's theory can obviously be applied to this system.

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